Chap 2: Conformal group

[State conf grp/alg first]

2.1. Conformal compactification of $\mathbb{R}^{p,q}$

Recall: For $n = p+q > 2$, $\langle x \rangle \equiv g(x,x)$

then

$$\exists\ \iota : \mathbb{R}^{p,q} \hookrightarrow \mathbb{RP}^{n+1}$$

$$\underset{x}{\overset{\parallel\!|}{(x^1,\dots,x^n)}} \mapsto \left(\frac{1-\langle x \rangle}{2} : x^1 : \dots : x^n : \frac{1+\langle x \rangle}{2} \right)$$

where $\mathbb{RP}^{n+1} = \mathbb{R}^{n+2} \setminus \{0\} / \sim$

$$\xi \sim \xi' \Leftrightarrow \xi = \lambda \xi',\ \lambda \in \mathbb{R} \setminus \{0\},$$

Def: ! Quick rmk on quadric surfaces

$$\overline{\iota(\mathbb{R}^{p,q})} \equiv N_{p,q} = \{ (\xi^0 : \dots : \xi^{n+1}) \mid \langle \xi \rangle = 0 \}$$

(quadric of $\mathbb{RP}^{n+1}$)

Lem: Let $j : \mathbb{R}^{p+1,q+1} \twoheadrightarrow S^p \times S^q$ be the canonical proj

then
$$\pi \equiv \gamma|_{S^p \times S^q} : S^p \times S^q \to N^{p,q} \in \text{Smth}$$
is a double cover.

! This gives $N_{p,q} \in$ Riem induced from $S^{p,q}$

Def: $\tau : \mathbb{R}^{p,q} \to S^{p,q} \in$ Riem, s/t
$$\tau(x) = \frac{1}{r(x)} \left(\frac{1 - \langle x \rangle}{2}, x^1, \dots, x^n, \frac{1 + \langle x \rangle}{2} \right)$$
for
$$r(x) = \frac{1}{2}\left(1 + 2 \sum_{j=1}^{n} (x^j)^2 + \langle x \rangle^2\right)^{1/2} > \frac{1}{2}$$

! $\iota = \pi \circ \tau$:

Prop: τ : conf embedding with conf factor $\Omega = r^{-1}$

Thm: $\psi = \psi_\Lambda : N^{p,q} \to N^{p,q}$.
s/t
ξ

$\psi(\xi^0 : \dots : \xi^{n+1}) = \gamma(\Lambda \xi)$
for $\Lambda \in O(p+1, q+1)$

then $\psi \in \mathrm{Conf}(N^{p,q})$

Also, $\psi_\Lambda = \psi_{\Lambda'} \Rightarrow \Lambda = \pm \Lambda'$

$\Rightarrow \psi : O(p+1, q+1) \longrightarrow \mathrm{Conf}(\mathbb{R}^{p,q})$ not inj

Prf: • ψ: well-defined.

For $\xi \in N^{p,q} \Leftrightarrow \xi \in \mathbb{RP}^{n+1}, \langle \xi \rangle = 0$

So $\langle \Lambda \xi \rangle \overset{!}{=} \langle \xi \rangle = 0$

! Λ: isometry $\in O(p+1, q+1)$

$\Rightarrow \gamma(\Lambda \xi) \in N^{p,q}$ [since this doesn't depend on choice of Λ]

• ψ: conf: For $P \in N^{p,q}$ (represented by $\xi \in S^{p,q}$

then ψ: conf with $\Omega^2(P) = \sum_{j=0}^{n+1} (\Lambda^j_k \xi^k)^2$

Def. • $\varphi: M \to \mathbb{R}^{p,q}$: conf

$\hat{\varphi}: N^{p,q} \to N^{p,q}$: conf continuation of φ

if $\hat{\varphi} \in \mathrm{Conf}(N^{p,q})$ and

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\iota"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R}^{p,q} \arrow[d, "\iota"] \\ N^{p,} \arrow[r, "\hat{\varphi}"] & N^{p,q} \end{tikzcd}$$

• $N^{p,q}$: conf cptification of $\mathbb{R}^{p,q}$

! We (have been) abuse by setting

$\varphi \in \mathrm{Conf}(N^{p,q})$

!• In general, $X \in \mathrm{Riem}^{\mathrm{conn}}$

N : conf cptification of X if

$\exists\ \iota : X \to N$: conf embedding s.t.

1, $\iota(X) \underset{\text{conn}}{\subseteq} N$ dense.

2, $\forall\ \varphi : X \supseteq U \to X$: conf

$\exists\ \hat{\varphi} : N \to N$: conf continuation

- For $X = \mathbb{R}^{p,q}$, $N = N^{p,q}$

$X \neq \mathbb{R}^{p,q}$ N : might not be cpt.
(X = unicover for AdS_n)

2.2 . $\mathrm{Conf}(\mathbb{R}^{p+q})$ for $p+q > 2$

Thm: $\mathrm{Conf}(N^{p,q}) \simeq O(p+1, q+1)/\mathbb{Z}_2$

$$\mathrm{Conf}(\mathbb{R}^{p,q}) = \begin{cases} SO(p+1, q+1) & \text{other} \\ SO(p+1, q+1)/\mathbb{Z}_2 & \text{if } -\mathrm{id} \in O_0(p+1, q+1) \end{cases}$$

$\overset{\text{III}}{\mathrm{Conf}_0(N^{p,q})}$

(E.g. p, q : odd)

Prf: We construct $\hat{\varphi}$ for $\forall \varphi : M \to \mathbb{R}^{p,q}$

- Orthogonal transformations: $\varphi(x) = \Lambda'(x)$

$$\varphi(x) = \Lambda' x \text{ for } \Lambda' \in O(p,q)$$

Let $\Lambda = \left(\begin{array}{c|c|c} 1 & & \\ \hline & \Lambda' & \\ \hline & & 1 \end{array}\right) \in O(p+1, q+1)$

since $\Lambda^T g \Lambda = g$

Let $\hat{\varphi} : N^{p,q} \to N^{p,q}$ s/t

$$\hat{\varphi}(\xi^0 : \ldots : \xi^{n+1}) = (\xi^0 : \Lambda' \xi : \xi) \text{ for } \xi \in \mathbb{R}^{p,q}$$

So for $x \in \mathbb{R}^{p,q}$

$$\hat{\varphi}(\iota(x)) = \left(\frac{1 - \langle x \rangle}{2} : \Lambda' x : \frac{1 + \langle x \rangle}{2} \right)$$

$$= \left(\frac{1 - \langle \Lambda' x \rangle}{2} : \Lambda' x : \frac{1 + \langle \Lambda' x \rangle}{2} \right)$$

$$\Rightarrow \hat{\varphi}(\iota(x)) = \iota(\varphi(x)) = \iota(\Lambda'(x)) \; \forall x$$

• Translation : $\varphi(x) = x + c$, $c \in \mathbb{R}^n$.

$$\hat{\varphi}(\xi^0 : \dots : \xi^{n+1}) =$$

$$(\xi^0 - \langle \xi', c \rangle - \xi^+ \langle c \rangle : \xi' + 2\xi^+ c : \xi^{n+1} + \langle \xi', c \rangle + \xi^+ \langle c \rangle)$$

for $\xi' \in \mathbb{R}^n$, $\xi^+ = \frac{1}{2}(\xi^{n+1} + \xi^0)$

$$\Rightarrow \hat{\varphi}(\iota(x))$$

$$= \left(\frac{1 - \langle x \rangle}{2} - \langle x, c \rangle - \frac{\langle c \rangle}{2} : x + c : \frac{1 + \langle x \rangle}{2} - \langle x, c \rangle - \frac{\langle c \rangle}{2} \right)$$

$$= \iota(\varphi(x))$$

Since $\langle x + c \rangle = \langle x \rangle + 2\langle x, c \rangle + \langle c \rangle$

Correspond to $\hat{\varphi} = \psi_{\Lambda_c}$ then we get

$$\Lambda_c = \begin{pmatrix} 1 - \frac{1}{2}\langle c \rangle & -(\eta' c)^T & -\frac{1}{2}\langle c \rangle \\ c & \mathrm{Id}_n & c \\ \frac{1}{2}\langle c \rangle & (\eta' c)^T & 1 + \frac{1}{2}\langle c \rangle \end{pmatrix}$$

for $\eta' = \mathrm{diag}(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q})$

$! \Lambda_c \in SO(p+1, q+1)$

- Dilation $\varphi(x) = \lambda x$

$$\Lambda_r = \begin{pmatrix} \frac{1+r^2}{2r} & 0 & \frac{1-r^2}{2r} \\ 0 & E_n & 0 \\ \frac{1-r^2}{2r} & 0 & \frac{1+r^2}{2r} \end{pmatrix} \in SO(p+1, q+1)$$

$\hat{\varphi} = \psi_{\Lambda_r}$

- SCT

$$\Lambda = \begin{pmatrix} 1 - \frac{1}{2}\langle b\rangle & -(\eta' b)^T & \frac{1}{2}\langle b\rangle \\ b & E_n & -b \\ -\frac{1}{2}\langle b\rangle & -(\eta' b)^T & 1 + \frac{1}{2}\langle b\rangle \end{pmatrix} \in \mathrm{SO}(p+1, q+1).$$

$\Rightarrow \forall \varphi : M \to \mathbb{R}^n$ can be extend

to $\hat{\varphi} : N^{p,q} \to N^{p,q}$ and $\hat{\varphi} = \psi_\Lambda$

is this surjective?

$$\psi : O(p+1, q+1) \longrightarrow \mathrm{Conf}(N^{p,q})$$
$$\pm\Lambda \longmapsto \psi_\Lambda$$

$\operatorname{Ker}\varphi = \mathbb{Z}_2 = \pm \mathrm{id}_{O(p+1,q+1)}$

$\Rightarrow O(p+1,q+1)/\mathbb{Z}_2 \underset{grp}{\simeq} \mathrm{Conf}(N^{p,q})$

$\Rightarrow \mathrm{Conf}(\mathbb{R}^{p,q}) = SO(p+1,q+1)$

2.3. $\mathrm{Conf}(\mathbb{R}^{2,0})$

Recall:

$$\{\varphi : M \to \mathbb{R}^{2,0} \mid \varphi : \text{conf}\} \simeq \mathcal{H}(\mathbb{C})^{\varphi' \neq 0}$$

But since there are a lot of

$\varphi \in \mathcal{H}(\mathbb{C})$ which is not inj

$(z \mapsto z^2)$

or DNE holomorphic $\hat{\varphi}$ $(z \mapsto z^{1/2})$ for $\mathrm{Re}\, z > 0$

So we need a refinement.

Def: $\varphi : M \to \mathbb{R}^{2,0}$: global conf trans

if $M = \mathbb{R}^{2,0}$, $\varphi \in \mathcal{H}(\mathbb{C})$ with at most 1 non-holomorphic point.

Thm: φ : global conf trans on $\mathbb{R}^{2,0}$ then $\exists!\, \hat{\varphi} : N^{2,0} \to N^{2,0}$ s.t. $\hat{\varphi} = \psi_\Lambda$ with $\Lambda \in O(3,1)$

$\mathrm{Conf}(N^{2,0}) \simeq O(3,1)/\mathbb{Z}_2$

$\mathrm{Conf}(\mathbb{R}^{2,0}) \simeq SO(3,1)$

! • For $p+q>2$ then φ: global conf if $M = \mathbb{R}^{p,q}$ or M_1

• $N^{2,0} \simeq S^2 \simeq \mathbb{C}P^1$ ≈ Riem sphere.

($N^{p,0} \simeq S^p$, since $S^{p,0} \simeq S^p \times \{\pm 1\}$ which is double cover of $N^{p,0}$)

• $\varphi_A : \mathbb{C} \to \mathbb{C}$: global conf.

$$z \mapsto \frac{az+b}{cz+d}$$

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$

$\Rightarrow \varphi : SL(2,\mathbb{C}) \longrightarrow \mathrm{Aut}(\mathbb{C})$

this induces

≈ Möbius grp

$PSL(2,\mathbb{C}) \simeq \mathrm{Aut}_{\mathrm{smth}}(\mathbb{C}P^1) \simeq \mathrm{Conf}(\mathbb{C}P^1) \simeq SO(3,1)$

2.4. $\dim \mathrm{Conf}(\mathbb{R}^{2,0}) = \infty$?

In physics they only care about Lie Alg and here $\mathrm{Lie}(\mathrm{Conf}(\mathbb{R}^{2,0})) = W$ ← Witt alg which is inf dim.

2.5. $\mathrm{Conf}(\mathbb{R}^{1,1})$

Recall: $\varphi = (u,v) : M \to \mathbb{R}^{1,1}$: conf

$\Leftrightarrow$

$$u_x = \pm v_y, \; u_y = \pm v_x, \quad u_x^2 > v_x^2$$

Thm: $f \in C^\infty(\mathbb{R})$, let $f_\pm \in C^\infty(\mathbb{R}^2, \mathbb{R})$ Light cone coord

by

$$f_\pm(x,y) = f(x \pm y) = f(x_\pm)$$

and $\Phi : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2, \mathbb{R}^2)$

$$(f,g) \mapsto \frac{1}{2}(f_+ + g_-, f_+ - g_-)$$

then 1, $\mathrm{Im}\,\Phi = \{(u,v) \mid u_x = v_y, u_y = v_x\}$

2, $\Phi(f,g)$: conf $\Leftrightarrow$ $f', g' > 0$ or

$f', g' < 0$

3, $\Phi(f,g)$: iso $\Leftrightarrow$ f, g: iso

4, $\Phi(f \circ h, g \circ k) = \Phi(f,g) \circ \Phi(h,k)$

pf: Set $\Phi(f,g) = (u,v)$

1, It's clear that

$\operatorname{im} \Phi \subseteq \{(u,v) \mid u_x = v_y,\ u_y = v_x\}$

Conversely, if $u_x = v_y,\ u_y = v_x$

$\Rightarrow u_{xx} = v_{yx} = u_{yy} \Rightarrow u(x,y)$: satisfies wave eq

$\Rightarrow u(x,y) = \frac{1}{2}(f_+(x,y) + g_-(x,y))$

2, $u_x^2 - v_x^2 = f_+' g_-' > 0 \Leftrightarrow f'g' > 0$

4, 3, Easy check

Cor: $\mathrm{Conf}(\mathbb{R}^{1,1})$

$$\mathrm{IS}$$

$$\left(\mathrm{Diff}_+(\mathbb{R})\times\mathrm{Diff}_+(\mathbb{R})\right)\cup\left(\mathrm{Diff}_-(\mathbb{R})\times\mathrm{Diff}_-(\mathbb{R})\right)$$

orient preserving. reversing.

$$\Phi(f,g) \leftarrow (f,g)$$

$$\varphi=(u,v)\mapsto(u,v)$$

Cor: $\mathrm{Conf}_0(S^{1,1})\simeq\mathrm{Diff}_+(S^1)\times\mathrm{Diff}_+(S^1)$

! $\mathrm{Lie}(\mathrm{Diff}_+(S^1))=\mathfrak{X}(S^1)$

$$\mathfrak{X}(S^1)_{\mathbb{C}}\simeq W$$

! $SO(2,2)/\mathbb{Z}_2\subsetneq\mathrm{Conf}(S^{1,1})$

Restricted conf grp

this is generated by translations, Lorentz trans, SCT,

Prop.

$$SO(2,2)/\mathbb{Z}_2 \simeq PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$$